Formulas in Solid Mechanics

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This collection of formulas is intended for use by foreign students in the course TMHL61, Damage Mechanics and Life Analysis, as a complement to the textbook Dahlberg and Ekberg: Failure, Fracture, Fatigue - An Introduction, Studentlitteratur, Lund, Sweden, 2002. It may be use at examinations in this course.

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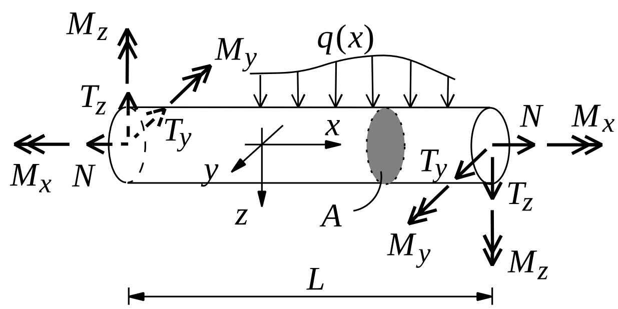
Version 03-09-18

1. Definitions and notations

Definition of coordinate system and loadings on beam



Loaded beam, length *L*, cross section *A*, and load $q(x)$, with coordinate system (origin at the geometric centre of cross section) and positive section forces and moments: normal force *N*, shear forces T_y and T_z, torque M_x, and bending moments M_y, M_z

Notations

Quantity	Symbol	SI Unit
Coordinate directions, with origin at geometric centre of cross-sectional area *A*	x, y, z	m
Normal stress in direction i (= x, y, z)	σ_i	N/m^2
Shear stress in direction j on surface with normal direction i	τ_{ij}	N/m^2
Normal strain in direction i	ε_i	–
Shear strain (corresponding to shear stress τ_{ij})	γ_{ij}	rad
Moment with respect to axis i	M, M_i	Nm
Normal force	N, P	N (= kg m/s^2)
Shear force in direction i (= y, z)	T, T_i	N
Load	$q(x)$	N/m
Cross-sectional area	A	m^2
Length	L, L_0	m
Change of length	δ	m
Displacement in direction x	u, $u(x)$, $u(x,y)$	m
Displacement in direction y	v, $v(x)$, $v(x,y)$	m
Beam deflection	$w(x)$	m
Second moment of area (i = y, z)	I, I_i	m^4
Modulus of elasticity (Young's modulus)	E	N/m^2
Poisson's ratio	ν	–
Shear modulus	G	N/m^2
Bulk modulus	K	N/m^2
Temperature coefficient	α	K^{-1}

2. Stress, Strain, and Material Relations

Normal stress σ_x

$$\sigma_x = \frac{N}{A} \quad \text{or} \quad \sigma_x = \lim_{\Delta A \to 0}\left(\frac{\Delta N}{\Delta A}\right)$$

ΔN = fraction of normal force N
ΔA = cross-sectional area element

Shear stress τ_{xy} (mean value over area A in the y direction)

$$\tau_{xy} = \frac{T_y}{A} \; (= \tau_{\text{mean}})$$

Normal strain ε_x

Linear, at small deformations ($\delta << L_0$)

$$\varepsilon_x = \frac{\delta}{L_0} \quad \text{or} \quad \varepsilon_x = \frac{\mathrm{d}u(x)}{\mathrm{d}x}$$

δ = change of length
L_0 = original length
$u(x)$ = displacement

Non-linear, at large deformations

$$\varepsilon_x = \ln\left(\frac{L}{L_0}\right)$$

L = actual length ($L = L_0 + \delta$)

Shear strain γ_{xy}

$$\gamma_{xy} = \frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial x}$$

Linear elastic material (Hooke's law)

Tension/compression

$$\varepsilon_x = \frac{\sigma_x}{E} + \alpha \Delta T$$

ΔT = *change* of temperatur

Lateral strain

$$\varepsilon_y = -\nu\, \varepsilon_x$$

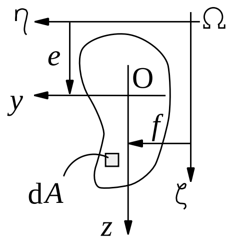
Shear strain

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

Relationships between *G*, *K*, *E* and ν

$$G = \frac{E}{2(1+\nu)} \qquad K = \frac{E}{3(1-2\nu)}$$

3. Geometric Properties of Cross-Sectional Area



The origin of the coordinate system Oyz is at the geometric centre of the cross section

Cross-sectional area A

$$A = \int_A \mathrm{d}A$$

dA = area element

Geometric centre (centroid)

$$e \cdot A = \int_A \zeta \,\mathrm{d}A$$

$$f \cdot A = \int_A \eta \,\mathrm{d}A$$

$e = \zeta_{gc}$ = distance from η axis to geometric centre

$f = \eta_{gc}$ distance from ζ axis to geometric centre

First moment of area

$$S_y = \int_{A'} z\,\mathrm{d}A \quad \text{and} \quad S_z = \int_{A'} y\,\mathrm{d}A$$

A' = the "sheared" area (part of area A)

Second moment of area

$$I_y = \int_A z^2 \mathrm{d}A$$

I_y = second moment of area with respect to the y axis

$$I_z = \int_A y^2 \mathrm{d}A$$

I_z = second moment of area with respect to the z axis

$$I_{yz} = \int_A yz\,\mathrm{d}A$$

I_{yz} = second moment of area with respect to the y and z axes

Parallel-axis theorems

First moment of area

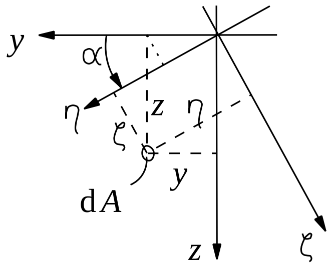
$$S_\eta = \int_A (z + e)\,\mathrm{d}A = eA \qquad \text{and} \qquad S_\zeta = \int_A (y + f)\,\mathrm{d}A = fA$$

Second moment of area

$$I_\eta = \int_A (z + e)^2\,\mathrm{d}A = I_y + e^2 A\,, \qquad I_\zeta = \int_A (y + f)^2\,\mathrm{d}A = I_z + f^2 A\,,$$

$$I_{\eta\zeta} = \int_A (z + e)(y + f)\,\mathrm{d}A = I_{yz} + efA$$

Rotation of axes



Coordinate system Ωηζ has been rotated the angle α with respect to the coordinate system O*yz*

$$I_\eta = \int_A \zeta^2 \,\mathrm{d}A = I_y \cos^2\alpha + I_z \sin^2\alpha - 2I_{yz} \sin\alpha \cos\alpha$$

$$I_\zeta = \int_A \eta^2 \,\mathrm{d}A = I_y \sin^2\alpha + I_z \cos^2\alpha + 2I_{yz} \sin\alpha \cos\alpha$$

$$I_{\eta\zeta} = \int_A \zeta\eta \,\mathrm{d}A = (I_y - I_z) \sin\alpha \cos\alpha + I_{yz} (\cos^2\alpha - \sin^2\alpha) = \frac{I_y - I_z}{2} \sin 2\alpha + I_{yz} \cos 2\alpha$$

Principal moments of area

$$I_{1,2} = \frac{I_y + I_z}{2} \pm R \quad \text{where} \quad R = \sqrt{\left(\frac{I_y - I_z}{2}\right)^2 + I_{yz}^2}$$

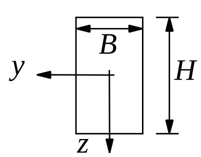
$$I_1 + I_2 = I_y + I_z$$

Principal axes

$$\sin 2\alpha = \frac{-I_{yz}}{R} \quad \text{or} \quad \cos 2\alpha = \frac{I_y - I_z}{2R}$$

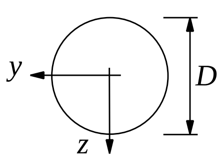
A line of symmetry is always a principal axis

Second moment of area with respect to axes through geometric centre for some symmetric areas (beam cross sections)



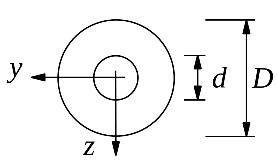
Rectangular area, base *B*, height *H*

$$I_y = \frac{BH^3}{12} \quad \text{and} \quad I_z = \frac{HB^3}{12}$$



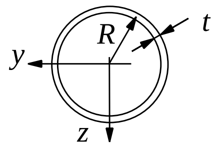
Solid circular area, diameter *D*

$$I_y = I_z = \frac{\pi D^4}{64}$$



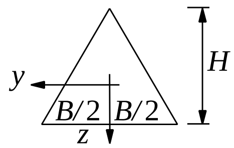
Thick-walled circular tube, diameters *D* and *d*

$$I_y = I_z = \frac{\pi}{64} (D^4 - d^4)$$



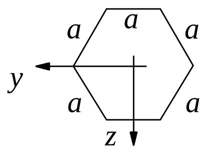
Thin-walled circular tube, radius R and wall thickness t ($t << R$)

$$I_y = I_z = \pi R^3 t$$



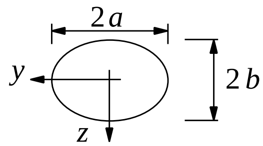
Triangular area, base B and height H

$$I_y = \frac{BH^3}{36} \quad \text{and} \quad I_z = \frac{HB^3}{48}$$



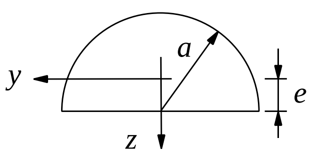
Hexagonal area, side length a

$$I_y = I_z = \frac{5\sqrt{3}}{16} a^4$$



Elliptical area, major axis $2a$ and minor axis $2b$

$$I_y = \frac{\pi a b^3}{4} \quad \text{and} \quad I_z = \frac{\pi b a^3}{4}$$



Half circle, radius a (geometric centre at e)

$$I_y = \left(\frac{\pi}{8} - \frac{8}{9\pi}\right) a^4 \cong 0,110\, a^4 \quad \text{and} \quad e = \frac{4a}{3\pi}$$

4. One-Dimensional Bodies (bars, axles, beams)

Tension/compression of bar

Change of length

$$\delta = \frac{NL}{EA} \quad \text{or}$$

N, E, and A are constant along bar
L = length of bar

$$\delta = \int_0^L \varepsilon(x)\mathrm{d}x = \int_0^L \frac{N(x)}{E(x)A(x)}\mathrm{d}x$$

$N(x)$, $E(x)$, and $A(x)$ may vary along bar

Torsion of axle

Maximum shear stress

$$\tau_{\max} = \frac{M_\mathrm{v}}{W_\mathrm{v}}$$

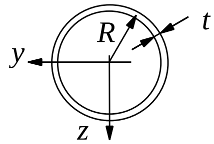
M_v = torque = M_x
W_v = section modulus in torsion (given below)

Torsion (deformation) angle

$$\Theta = \frac{M_\mathrm{v} L}{G K_\mathrm{v}}$$

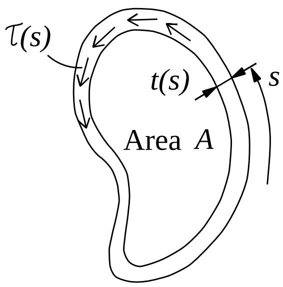
M_v = torque = M_x
K_v = section factor of torsional stiffness (given below)

Section modulus W_v and section factor K_v for some cross sections (at torsion)



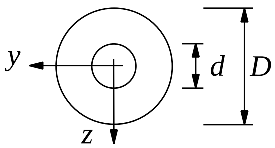
Torsion of thin-walled circular tube, radius R, thickness t, where $t << R$,

$W_v = 2\pi R^2 t \qquad K_v = 2\pi R^3 t$



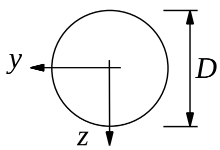
Thin-walled tube of arbitrary cross section
A = area enclosed by the tube
$t(s)$ = wall thickness
s = coordinate around the tube

$$W_v = 2A t_{min} \qquad K_v = \frac{4A^2}{\oint_s [t(s)]^{-1}\, ds}$$



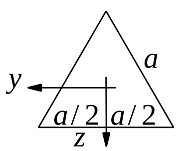
Thick-walled circular tube, diameters D and d,

$$W_v = \frac{\pi}{16}\frac{D^4 - d^4}{D} \qquad K_v = \frac{\pi}{32}(D^4 - d^4)$$



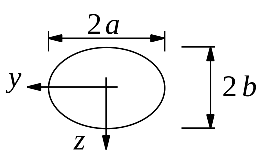
Solid axle with circular cross section, diameter D,

$$W_v = \frac{\pi D^3}{16} \qquad K_v = \frac{\pi D^4}{32}$$



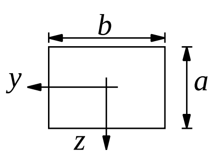
Solid axle with triangular cross section, side length a

$$W_v = \frac{a^3}{20} \qquad K_v = \frac{a^4\sqrt{3}}{80}$$



Solid axle with elliptical cross section, major axle $2a$ and minor axle $2b$

$$W_v = \frac{\pi}{2} a\, b^2 \qquad K_v = \frac{\pi a^3 b^3}{a^2 + b^2}$$



Solid axle with rectangular cross section b by a, where $b \geq a$

$W_v = k_{Wv}\, a^2 b \qquad K_v = k_{Kv}\, a^3 b$

for k_{Wv} and k_{Kv}, see table below

Factors k_{Wv} and k_{Kv} for some values of ratio b / a (solid rectangular cross section)

b/a	k_{Wv}	k_{Kv}
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

Bending of beam

Relationships between bending moment $M_y = M(x)$, shear force $T_z = T(x)$, and load $q(x)$ on beam

$$\frac{\mathrm{d}T(x)}{\mathrm{d}x} = -q(x), \qquad \frac{\mathrm{d}M(x)}{\mathrm{d}x} = T(x), \quad \text{and} \quad \frac{\mathrm{d}^2M(x)}{\mathrm{d}x^2} = -q(x)$$

Normal stress

$$\sigma = \frac{N}{A} + \frac{Mz}{I}$$

I (here I_y) = second moment of area (see Section 12.2)

Maximum bending stress

$$|\sigma|_{\max} = \frac{|M|}{W_b} \quad \text{where} \quad W_b = \frac{I}{|z|_{\max}}$$

W_b = section modulus (in bending)

Shear stress

$$\tau = \frac{TS_{A'}}{Ib}$$

$S_{A'}$ = first moment of area A' (see Section 12.2)

b = length of line limiting area A'

$$\tau_{gc} = \mu \frac{T}{A}$$

τ_{gc} = shear stress at geometric centre

μ = the Jouravski factor

The Jouravski factor μ for some cross sections

rectangular	1.5
triangular	1.33
circular	1.33
thin-walled circular	2.0
elliptical	1.33
ideal I profile	A / A_{web}

Skew bending

Axes y and z are not principal axes:

$$\sigma = \frac{N}{A} + \frac{M_y(zI_z - yI_{yz}) - M_z(yI_y - zI_{yz})}{I_y\, I_z - I_{yz}^2}$$

I_y, I_z, I_{yz} = second moment of area

Axes y' and z' are principal axes:

$$\sigma = \frac{N}{A} + \frac{M_1\, z'}{I_1} - \frac{M_2\, y'}{I_2}$$

I_1, I_2 = principal second moment of area

Beam deflection $w(x)$

Differential equations

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left\{ EI(x)\, \frac{\mathrm{d}^2}{\mathrm{d}x^2} w(x) \right\} = q(x)$$

when $EI(x)$ is function of x

$$EI\, \frac{\mathrm{d}^4}{\mathrm{d}x^4} w(x) = q(x)$$

when EI is constant

Homogeneous boundary conditions

Clamped beam end

$$w(*) = 0 \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x} w(*) = 0$$

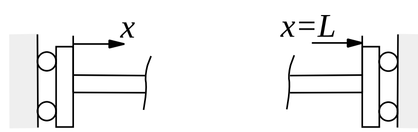
where * is the coordinate of beam end
(to be entered after differentiation)

Simply supported beam end

$$w(*) = 0 \quad \text{and} \quad -EI\, \frac{\mathrm{d}^2}{\mathrm{d}x^2} w(*) = 0$$

Sliding beam end

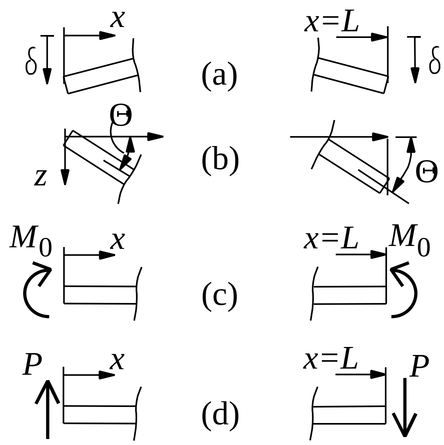
$$\frac{\mathrm{d}}{\mathrm{d}x} w(*) = 0 \quad \text{and} \quad -EI\, \frac{\mathrm{d}^3}{\mathrm{d}x^3} w(*) = 0$$



Free beam end

$$-EI\, \frac{\mathrm{d}^2}{\mathrm{d}x^2} w(*) = 0 \quad \text{and} \quad -EI\, \frac{\mathrm{d}^3}{\mathrm{d}x^3} w(*) = 0$$

Non-homogeneous boundary conditions



(a) Displacement δ prescribed

$$w(*) = \delta$$

(b) Slope Θ prescribed

$$\frac{\mathrm{d}}{\mathrm{d}x} w(*) = \Theta$$

(c) Moment M_0 prescribed

$$-EI \frac{\mathrm{d}^2}{\mathrm{d}x^2} w(*) = M_0$$

(d) Force P prescribed

$$-EI \frac{\mathrm{d}^3}{\mathrm{d}x^3} w(*) = P$$

Beam on elastic bed

Differential equation

$$EI \frac{\mathrm{d}^4}{\mathrm{d}x^4} w(x) + k w(x) = q(x)$$

EI = constant bending stiffness
k = bed modulus (N/m^2)

Solution

$$w(x) = w_{\text{part}}(x) + w_{\text{hom}}(x) \quad \text{where}$$

$$w_{\text{hom}}(x) = \{C_1 \cos(\lambda x) + C_2 \sin(\lambda x)\}\ \mathrm{e}^{\lambda x} + \{C_3 \cos(\lambda x) + C_2 \sin(\lambda x)\}\ \mathrm{e}^{-\lambda x}\ ; \quad \lambda^4 = \frac{k}{4EI}$$

Boundary conditions as given above

Beam vibration

Differential equation

$$EI \frac{\partial^4}{\partial x^4} w(x,t) + m \frac{\partial^2}{\partial t^2} w(x,t) = q(x,t)$$

EI = constant bending stiffness
m = beam mass per metre (kg/m)
t = time

Assume solution $w(x,t) = X(x)\cdot T(t)$. Then the standing wave solution is

$$T(t) = e^{i\omega t} \quad \text{and} \quad X(x) = C_1 \cosh(\mu x) + C_2 \cos(\mu x) + C_3 \sinh(\mu x) + C_4 \sin(\mu x)$$

where $\mu^4 = \omega^2 m / EI$

Boundary conditions (as given above) give an eigenvalue problem that provides the eigenfrequencies and eigenmodes (eigenforms) of the vibrating beam

Axially loaded beam, stability, the Euler cases

Beam axially loaded in tension

Differential equation

$$EI\,\frac{\mathrm{d}^4}{\mathrm{d}x^4}w(x) - N\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}w(x) = q(x) \qquad N = \text{normal force in tension } (N > 0)$$

Solution

$w(x) = w_{\text{part}}(x) + w_{\text{hom}}(x)$ where

$$w_{\text{hom}}(x) = C_1 + C_2\sqrt{\frac{N}{EI}}\,x + C_3\sinh\left(\sqrt{\frac{N}{EI}}\,x\right) + C_4\cosh\left(\sqrt{\frac{N}{EI}}\,x\right)$$

New boundary condition on shear force (other boundary conditions as given above)

$$T(*) = -EI\,\frac{\mathrm{d}^3}{\mathrm{d}x^3}w(*) + N\,\frac{\mathrm{d}}{\mathrm{d}x}w(*)$$

Beam axially loaded in compression

Differential equation

$$EI\,\frac{\mathrm{d}^4}{\mathrm{d}x^4}w(x) + P\,\frac{\mathrm{d}^2}{\mathrm{d}x^2}w(x) = q(x) \qquad P = \text{normal force in compression } (P > 0)$$

Solution

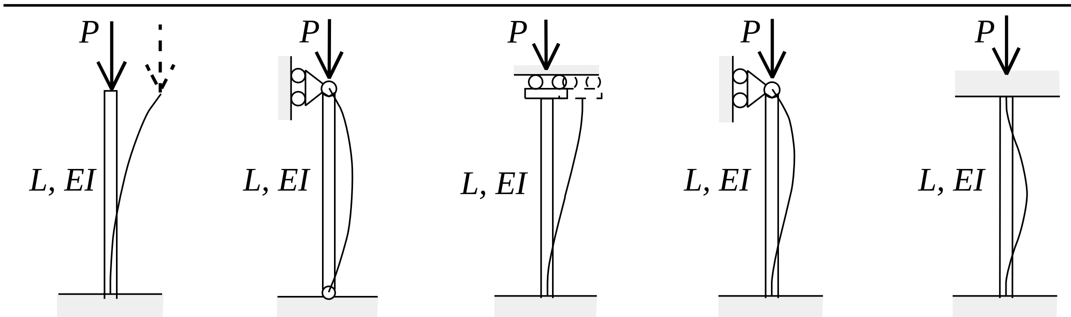
$w(x) = w_{\text{part}}(x) + w_{\text{hom}}(x)$ where

$$w_{\text{hom}}(x) = C_1 + C_2\sqrt{\frac{P}{EI}}\,x + C_3\sin\left(\sqrt{\frac{P}{EI}}\,x\right) + C_4\cos\left(\sqrt{\frac{P}{EI}}\,x\right)$$

New boundary condition on shear force (other boundary conditions as given above)

$$T(*) = -EI\,\frac{\mathrm{d}^3}{\mathrm{d}x^3}w(*) - P\,\frac{\mathrm{d}}{\mathrm{d}x}w(*)$$

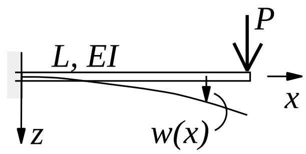
Elementary cases: the Euler cases (P_c is critical load)



Case 1	Case 2a	Case 2b	Case 3	Case 4
$P_c = \dfrac{\pi^2 EI}{4L^2}$	$P_c = \dfrac{\pi^2 EI}{L^2}$	$P_c = \dfrac{\pi^2 EI}{L^2}$	$P_c = \dfrac{2.05\,\pi^2 EI}{L^2}$	$P_c = \dfrac{4\pi^2 EI}{L^2}$

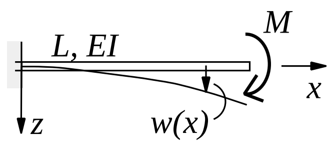
5. Bending of Beam – Elementary Cases

Cantilever beam



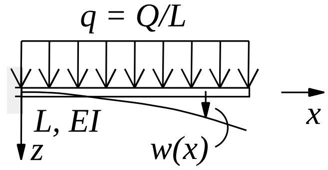
$$w(x) = \frac{PL^3}{6EI}\left(3\frac{x^2}{L^2} - \frac{x^3}{L^3}\right)$$

$$w(L) = \frac{PL^3}{3EI} \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = \frac{PL^2}{2EI}$$



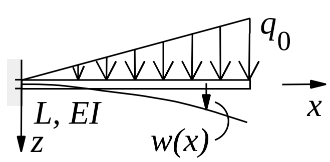
$$w(x) = \frac{ML^2}{2EI}\left(\frac{x^2}{L^2}\right)$$

$$w(L) = \frac{ML^2}{2EI} \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = \frac{ML}{EI}$$



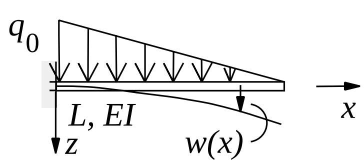
$$w(x) = \frac{qL^4}{24EI}\left(\frac{x^4}{L^4} - 4\frac{x^3}{L^3} + 6\frac{x^2}{L^2}\right)$$

$$w(L) = \frac{qL^4}{8EI} \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = \frac{qL^3}{6EI}$$



$$w(x) = \frac{q_0 L^4}{120EI}\left(\frac{x^5}{L^5} - 10\frac{x^3}{L^3} + 20\frac{x^2}{L^2}\right)$$

$$w(L) = \frac{11\, q_0 L^4}{120EI} \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = \frac{q_0 L^3}{8EI}$$

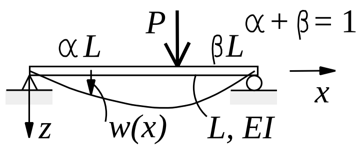


$$w(x) = \frac{q_0 L^4}{120EI}\left(-\frac{x^5}{L^5} + 5\frac{x^4}{L^4} - 10\frac{x^3}{L^3} + 10\frac{x^2}{L^2}\right)$$

$$w(L) = \frac{q_0 L^4}{30EI} \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = \frac{q_0 L^3}{24EI}$$

Simply supported beam

Load applied at $x = \alpha L$ ($\alpha < 1$), $\beta = 1 - \alpha$

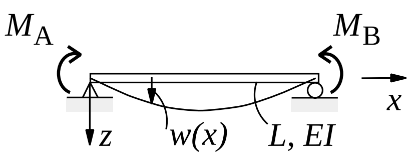


$$w(x) = \frac{PL^3}{6EI}\,\beta\left((1-\beta^2)\frac{x}{L} - \frac{x^3}{L^3}\right) \quad \text{for} \quad \frac{x}{L} \le \alpha$$

$$w(\alpha L) = \frac{PL^3}{3EI}\,\alpha^2\beta^2\,. \quad \text{When } \alpha > \beta \text{ one obtains}$$

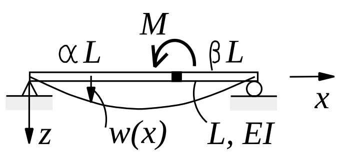
$$w_{\max} = w\left(L\sqrt{\frac{1-\beta^2}{3}}\right) = w(\alpha L)\,\frac{1+\beta}{3\beta}\sqrt{\frac{1+\beta}{3\alpha}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}w(0) = \frac{PL^2}{6EI}\,\alpha\beta(1+\beta) \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = -\frac{PL^2}{6EI}\,\alpha\beta(1+\alpha)$$



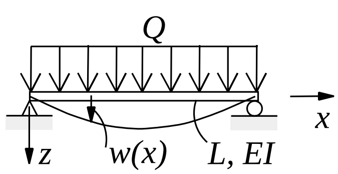
$$w(x) = \frac{L^2}{6EI}\left\{M_A\left(2\frac{x}{L} - 3\frac{x^2}{L^2} + \frac{x^3}{L^3}\right) + M_B\left(\frac{x}{L} - \frac{x^3}{L^3}\right)\right\}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}w(0) = \frac{M_A L}{3\,EI} + \frac{M_B L}{6\,EI} \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = -\frac{M_A L}{6\,EI} - \frac{M_B L}{3\,EI}$$



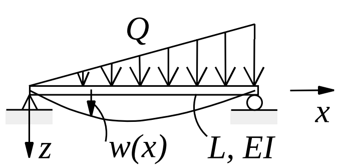
$$w(x) = \frac{ML^2}{6EI}\left((1-3\beta^2)\frac{x}{L} - \frac{x^3}{L^3}\right) \quad \text{for} \quad \frac{x}{L} \le \alpha$$

$$\frac{\mathrm{d}}{\mathrm{d}x}w(0) = \frac{ML}{6EI}(1-3\beta^2) \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = \frac{ML}{6EI}(1-3\alpha^2)$$



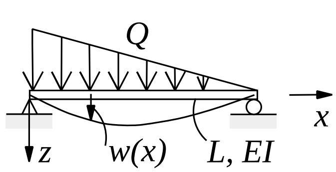
$$w(x) = \frac{QL^3}{24EI}\left(\frac{x^4}{L^4} - 2\frac{x^3}{L^3} + \frac{x}{L}\right)$$

$$w(L/2) = \frac{5\,QL^3}{384\,EI} \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(0) = -\frac{\mathrm{d}}{\mathrm{d}x}w(L) = \frac{QL^2}{24EI}$$



$$w(x) = \frac{QL^3}{180EI}\left(3\frac{x^5}{L^5} - 10\frac{x^3}{L^3} + 7\frac{x}{L}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}w(0) = \frac{7\,QL^2}{180EI} \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = -\frac{8\,QL^2}{180EI}$$



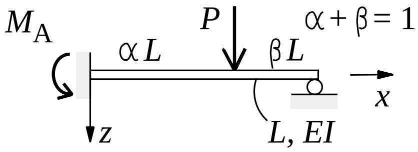
$$w(x) = \frac{QL^3}{180EI}\left(-3\frac{x^5}{L^5} + 15\frac{x^4}{L^4} - 20\frac{x^3}{L^3} + 8\frac{x}{L}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}w(0) = \frac{8\,QL^2}{180EI} \qquad \frac{\mathrm{d}}{\mathrm{d}x}w(L) = -\frac{7\,QL^2}{180EI}$$

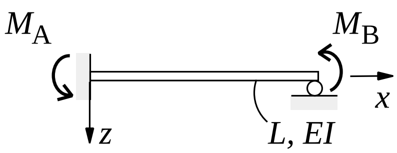
Clamped – simply supported beam and clamped – clamped beam

Load applied at $x = \alpha L$ ($\alpha < 1$), $\beta = 1 - \alpha$

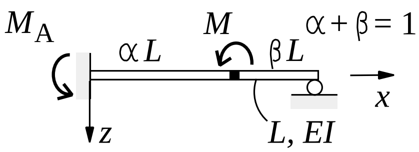
Only redundant reactions are given. For deflections, use superposition of solutions for simply supported beams.



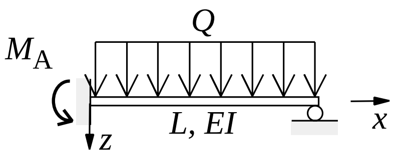
$$M_A = \frac{PL}{2}\beta(1-\beta^2)$$



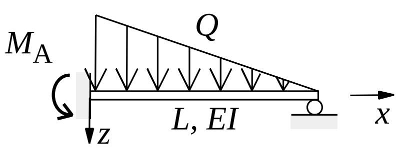
$$M_A = \frac{M_B}{2}$$



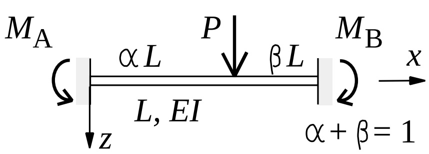
$$M_A = \frac{M}{2}(1-3\beta^2)$$



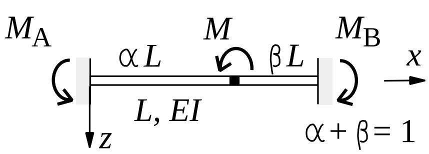
$$M_A = \frac{QL}{8}$$



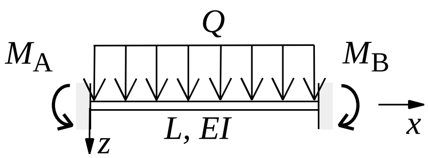
$$M_A = \frac{2\,QL}{15}$$



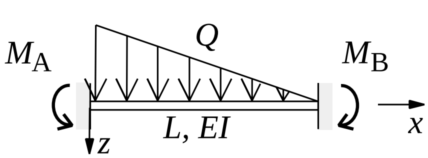
$$M_A = PL\,\alpha\beta^2 \qquad M_B = PL\,\alpha^2\beta$$



$$M_A = -M\,\beta(1-3\alpha) \qquad M_B = M\,\alpha(1-3\beta)$$



$$M_A = M_B = \frac{QL}{12}$$



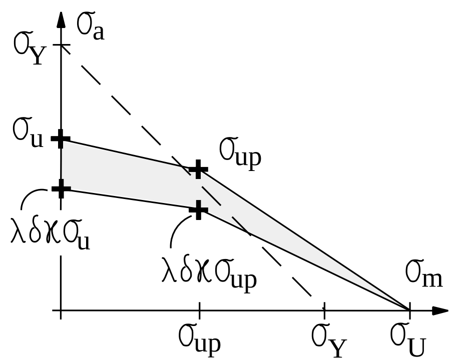
$$M_A = \frac{QL}{10} \qquad M_B = \frac{QL}{15}$$

6. Material Fatigue

Fatigue limits (notations)

Load	Alternating	Pulsating
Tension/compression	$\pm\sigma_u$	$\sigma_{up} \pm \sigma_{up}$
Bending	$\pm\sigma_{ub}$	$\sigma_{ubp} \pm \sigma_{ubp}$
Torsion	$\pm\tau_{uv}$	$\tau_{uvp} \pm \tau_{uvp}$

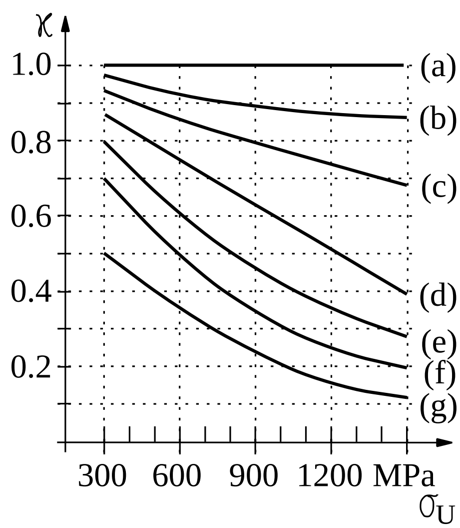
The Haigh diagram



σ_a = stress amplitude
σ_m = mean stress
σ_Y = yield limit
σ_U = ultimate strenght
σ_u, σ_{up} = fatigue limits
λ, δ, κ = factors reducing fatigue limits
(similar diagrams for σ_{ub}, σ_{ubp} and τ_{uv}, τ_{uvp})

Factors reducing fatigue limits

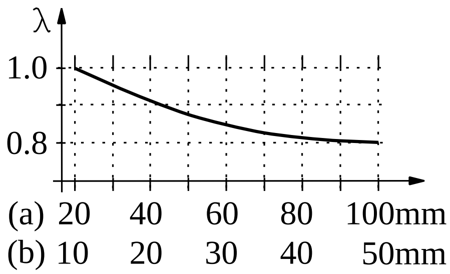
Surface finish κ



Factor κ reducing the fatigue limit due to surface irregularities

(a) polished surface (κ = 1)
(b) ground
(c) machined
(d) standard notch
(e) rolling skin
(f) corrosion in sweet water
(g) corrosion in salt water

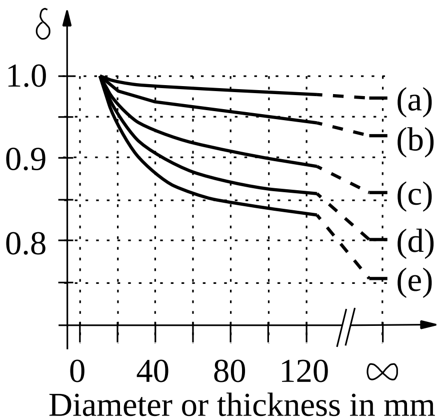
Volume factor λ (due to process)



Factor λ reducing the fatigue limit due to size of raw material

(a) diameter at circular cross section
(b) thickness at rectangular cross section

Volume factor δ (due to geometry)



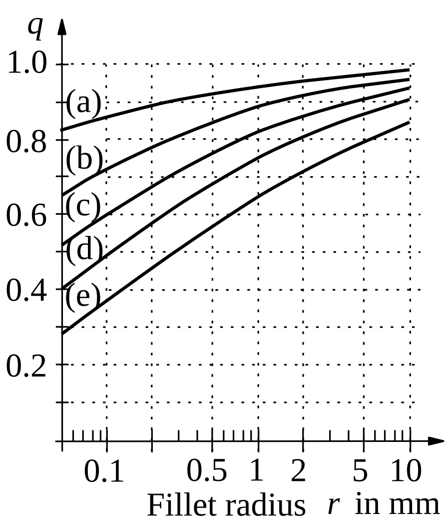
Factor δ reducing the fatigue limits σ_{ub} and τ_{uv} due to loaded volume.
Steel with ultimate strength σ_U =
(a) 1500 MPa
(b) 1000 MPa
(c) 600 MPa
(d) 400 MPa
(e) aluminium
Factor δ = 1 when fatigue notch factor K_f > 1 is used.

Fatigue notch factor K_f (at stress concentration)

$K_f = 1 + q\,(K_t - 1)$

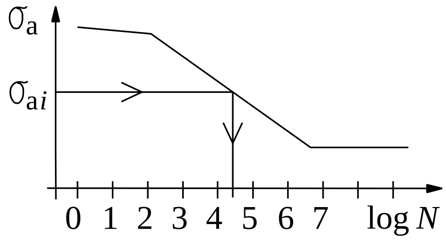
K_t = stress concentration factor (see Section 12.8)
q = fatigue notch sensitivity factor

Fatigue notch sensitivity factor *q*



Fatigue notch sensitivity factor q for steel with ultimate strength σ_U =
(a) 1600 MPa
(b) 1300 MPa
(c) 1000 MPa
(d) 700 MPa
(e) 400 MPa

Wöhler diagram



σ_{ai} = stress amplitude
N_i = fatigue life (in cycles) at stress amplitude σ_{ai}

Damage accumulation *D*

$$D = \frac{n_i}{N_i}$$

n_i = number of loading cycles at stress amplitude σ_{ai}
N_i = fatigue life at stress amplitude σ_{ai}

Palmgren-Miner's rule

Failure when

$$\sum_{i=1}^{I} \frac{n_i}{N_i} = 1$$

n_i = number of loading cycles at stress amplitude σ_{ai}
N_i = fatigue life at stress amplitude σ_{ai}
I = number of loading stress levels

Fatigue data (cyclic, constant-amplitude loading)

The following fatigue limits may be used *only* when solving exercises. For a real design, data should be taken from latest official standard and *not* from this table.[1]

Material	Tension alternating MPa	Tension pulsating MPa	Bending alternating MPa	Bending pulsating MPa	Torsion alternating MPa	Torsion pulsating MPa
Carbon steel						
141312-00	± 110	110 ± 110	± 170	150 ± 150	± 100	100 ± 100
141450-1	± 140	130 ± 130	± 190	170 ± 170	± 120	120 ± 120
141510-00	± 230					
141550-01	± 180	160 ± 160	± 240	210 ± 210	± 140	140 ± 140
141650-01	± 200	180 ± 180	± 270	240 ± 240	± 150	150 ± 150
141650			± 460			

Stainless steel 2337-02, $\sigma_u = \pm 270$ MPa

Aluminium SS 4120-02, $\sigma_{ub} = \pm 110$ MPa; SS 4425-06, $\sigma_u = \pm 120$ MPa

[1] Data in this table has been collected from B Sundström (editor): Handbok och Formelsamling i Hållfasthetslära, Institutionen för hållfasthetslära, KTH, Stockholm, 1998.

7. Multi-Axial Stress States

Stresses in thin-walled circular pressure vessel

$$\sigma_t = p\,\frac{R}{t} \quad \text{and} \quad \sigma_x = p\,\frac{R}{2t} \qquad (\sigma_z \approx 0)$$

σ_t = circumferential stress
σ_x = longitudinal stress
p = internal pressure
R = radius of pressure vessel
t = wall thickness ($t << R$)

Rotational symmetry in structure and load (plane stress, i.e. $\sigma_z = 0$)

Differential equation for rotating circular plate

$$\frac{d^2 u}{d r^2} + \frac{1}{r}\frac{d u}{d r} - \frac{u}{r^2} = -\frac{1-\nu^2}{E}\rho\omega^2 r$$

$u = u(r)$ = radial displacement
ρ = density
ω = angular rotation (rad/s)

Solution

$$u(r) = u_{\text{hom}} + u_{\text{part}} = A_0\,r + \frac{B_0}{r} - \frac{1-\nu^2}{8E}\rho\,\omega^2\,r^3$$

Stresses

$$\sigma_r(r) = A - \frac{B}{r^2} - \frac{3+\nu}{8}\rho\,\omega^2\,r^2 \qquad \text{and} \qquad \sigma_\phi(r) = A + \frac{B}{r^2} - \frac{1+3\nu}{8}\rho\,\omega^2\,r^2$$

where

$$A = \frac{E\,A_0}{1-\nu} \qquad \text{and} \qquad B = \frac{E\,B_0}{1+\nu}$$

Boundary conditions
σ_r or u must be known on inner and outer boundary of the circular plate

Shrink fit

$$\delta = u_{\text{outer}}(p) - u_{\text{inner}}(p)$$

δ = difference of radii
p = contact pressure
u = radial displacement as function of p

Plane stress and plane strain (plane state)

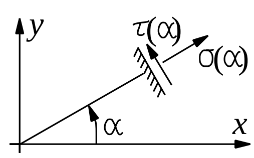
Plane stress (in xy-plane) when $\sigma_z = 0$, $\tau_{xz} = 0$, and $\tau_{yz} = 0$

Plane strain (in xy-plane) when $\tau_{xz} = 0$, $\tau_{yz} = 0$, and $\varepsilon_z = 0$ or constant

Stresses in direction α **(plane state)**

$$\sigma(\alpha) = \sigma_x \cos^2(\alpha) + \sigma_y \sin^2(\alpha) + 2\tau_{xy}\cos(\alpha)\sin(\alpha)$$

$$\tau(\alpha) = -(\sigma_x - \sigma_y)\sin(\alpha)\cos(\alpha) + \tau_{xy}(\cos^2(\alpha) - \sin^2(\alpha))$$



$\sigma(\alpha)$ = normal stress in direction α
$\tau(\alpha)$ = shear stress on surface with normal in direction α

Principal stresses $\sigma_{1,2}$ and principal directions at plane stress state

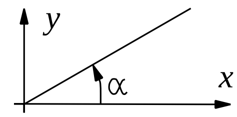
$$\sigma_{1,2} = \sigma_c \pm R = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sin(2\psi_1) = \frac{\tau_{xy}}{R} \quad \text{or} \quad \cos(2\psi_1) = \frac{\sigma_x - \sigma_y}{2R}$$

ψ_1 = angle from x axis (in xy plane) to direction of principal stress σ_1

Strain in direction α (plane state)

$$\varepsilon(\alpha) = \varepsilon_x \cos^2(\alpha) + \varepsilon_y \sin^2(\alpha) + \gamma_{xy} \sin(\alpha)\cos(\alpha)$$



$$\gamma(\alpha) = (\varepsilon_y - \varepsilon_x)\sin(2\alpha) + \gamma_{xy}\cos(2\alpha)$$

$\varepsilon(\alpha)$ = normal strain in direction α

$\gamma(\alpha)$ = shear strain of element with normal in direction α

Principal strains and principal directions (plane state)

$$\varepsilon_{1,2} = \varepsilon_c \pm R = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x - \varepsilon_y}{2}\right)^2 + \left(\frac{\gamma_{xy}}{2}\right)^2}$$

$$\sin(2\psi_1) = \frac{\gamma_{xy}}{2R} \quad \text{or} \quad \cos(2\psi_1) = \frac{\varepsilon_x - \varepsilon_y}{2R}$$

ψ_1 = angle from x axis (in xy plane) to direction of principal strain ε_1

Principal stresses and principal directions at three-dimensional stress state

The determinant

$|\,\mathbf{S} - \sigma\,\mathbf{I}\,| = 0$

gives three roots (the principal stresses)

Stress matrix $\mathbf{S} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$

(contains the nine stress components σ_{ij})

Unit matrix $\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Direction of principal stress σ_i (i = 1, 2, 3) is given by

$(\mathbf{S} - \sigma_i\,\mathbf{I}) \cdot \mathbf{n}_i = \mathbf{0}$

and

$\mathbf{n}_i^{\mathrm{T}} \cdot \mathbf{n}_i = 1$

n_{ix}, n_{iy} and n_{iz} are the elements of the unit vector $\mathbf{n}_i$ in the direction of σ_i

($^{\mathrm{T}}$ means transpose)

Principal strains and principal directions at three-dimensional stress state

Use shear strain $\varepsilon_{ij} = \gamma_{ij} / 2$ for $i \neq j$

The determinant

$| \mathbf{E} - \varepsilon \, \mathbf{I} | = 0$

gives three roots (the principal strains)

Strain matrix $\mathbf{E} = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$

$\mathbf{I}$ = unit matrix

Direction of principal strain ε_i (i = 1, 2, 3) is given by

$(\mathbf{E} - \varepsilon_i \, \mathbf{I}) \cdot \mathbf{n}_i = \mathbf{0}$

and

$\mathbf{n}_i^{\mathrm{T}} \cdot \mathbf{n}_i = 1$

n_{ix}, n_{iy} and n_{iz} are the elements of the unit vector $\mathbf{n}_i$ in the direction of ε_i

($^{\mathrm{T}}$ means transpose)

Hooke's law, including temperature term (three-dimensional stress state)

$$\varepsilon_{\mathrm{x}} = \frac{1}{E} [\sigma_{\mathrm{x}} - \nu(\sigma_{\mathrm{y}} + \sigma_z)] + \alpha \, \Delta T$$

$$\varepsilon_{\mathrm{y}} = \frac{1}{E} [\sigma_{\mathrm{y}} - \nu(\sigma_{\mathrm{z}} + \sigma_x)] + \alpha \, \Delta T$$

$$\varepsilon_{\mathrm{z}} = \frac{1}{E} [\sigma_{\mathrm{z}} - \nu(\sigma_{\mathrm{x}} + \sigma_y)] + \alpha \, \Delta T$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G} \qquad \gamma_{yz} = \frac{\tau_{yz}}{G} \qquad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

α = temperature coefficient

ΔT = *change* of temperature (relative to temperature giving no stress)

Effective stress

The Huber-von Mises effective stress (the deviatoric stress hypothesis)

$$\sigma_{\mathrm{e}}^{\mathrm{vM}} = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x\sigma_y - \sigma_y\sigma_z - \sigma_z\sigma_x + 3\tau_{xy}^2 + 3\tau_{yz}^2 + 3\tau_{zx}^2}$$

$$= \sqrt{\frac{1}{2}\{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\}}$$

The Tresca effective stress (the shear stress hypothesis)

$$\sigma_{\mathrm{e}}^{\mathrm{T}} = \max[\,|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\,] = \sigma_{\max}^{\mathrm{pr}} - \sigma_{\min}^{\mathrm{pr}}$$

(pr = principal stress)

8. Energy Methods – the Castigliano Theorem

Strain energy *u* per unit of volume

Linear elastic material and uni-axial stress

$$u = \frac{\sigma\,\varepsilon}{2}$$

Total strain energy *U* in beam loaded in tension/compression, torsion, bending, and shear

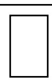
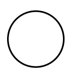
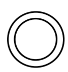
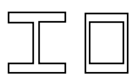
$$U_{\text{tot}} = \int_0^L \left\{ \frac{N(x)^2}{2EA(x)} + \frac{M_{\text{t}}(x)^2}{2GK_{\text{v}}(x)} + \frac{M_{\text{bend}}(x)^2}{2EI(x)} + \beta\,\frac{T(x)^2}{2GA(x)} \right\} \mathrm{d}x$$

M_{t} = torque = M_x

M_{bend} = bending moment = M_y

K_{v} = section factor of torsional stiffness

β = shear factor, see below

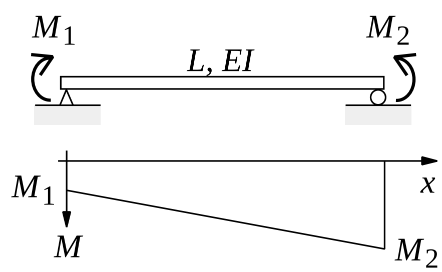
Cross section	β	μ
	6/5	3/2
	10/9	4/3
	2	2
	A/A_{web}	A/A_{web}

Shear factor β

$$\beta = \frac{A}{I^2}\int_A \left(\frac{S_{A'}}{b}\right)^2 \mathrm{d}A$$

β is given for some cross sections in the table (μ is the Jouravski factor, see Section 12.3 One-Dimensional Bodies)

Elementary case: pure bending



Only bending momentet M_{bend} is present. The moment varies linearly along the beam with moments M_1 and M_2 at the beam ends. One has

$M_{\text{bend}}(x) = M_1 + (M_2 - M_1)x/L$, which gives

$$U_{\text{tot}} = \frac{L}{6EI}\{M_1^2 + M_1 M_2 + M_2^2\}$$

The second term is negative if M_1 and M_2 have different signs

The Castigliano theorem

$$\delta = \frac{\partial U}{\partial P} \quad \text{and} \quad \Theta = \frac{\partial U}{\partial M}$$

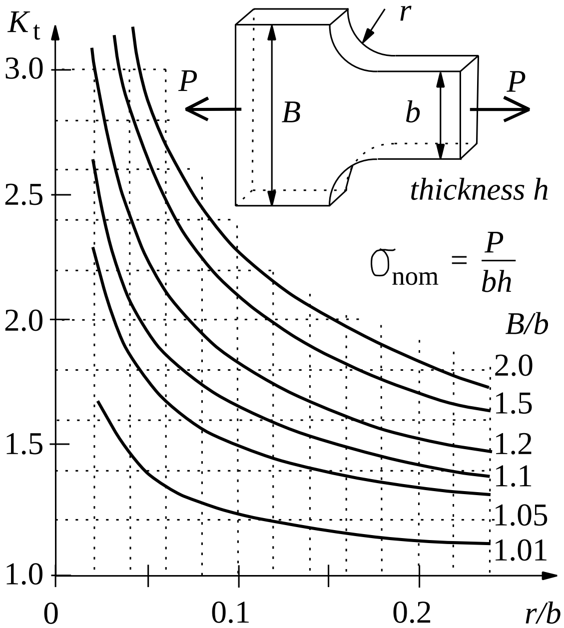
δ = displacement in the direction of force P of the point where force P is applied

Θ = rotation (change of angle) at moment M

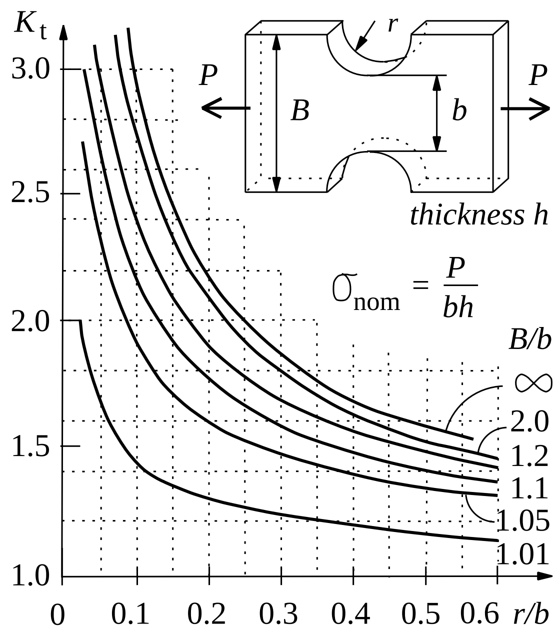
9. Stress Concentration

Tension/compression

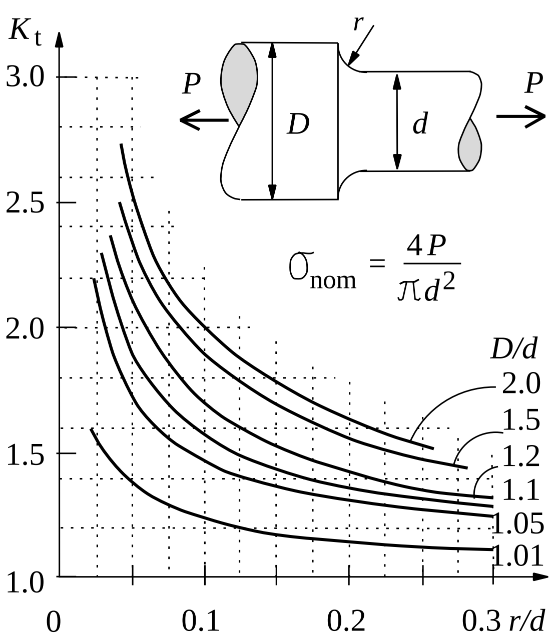
Maximum normal stress at a stress concentration is $\sigma_{max} = K_t \sigma_{nom}$, where K_t and σ_{nom} are given in the diagrams



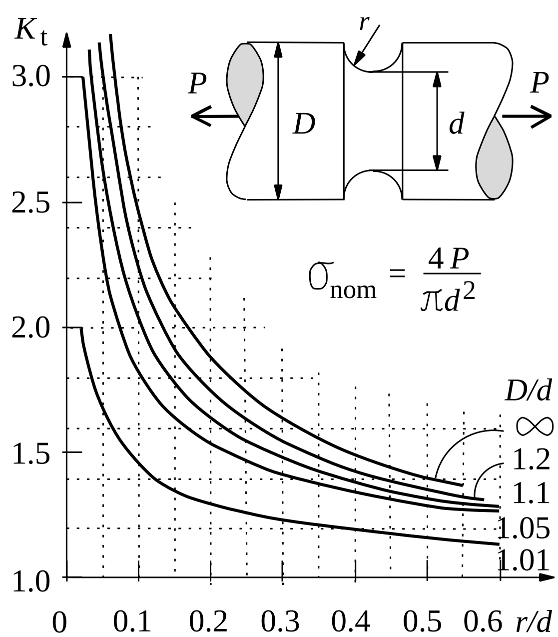
Tension of flat bar with shoulder fillet



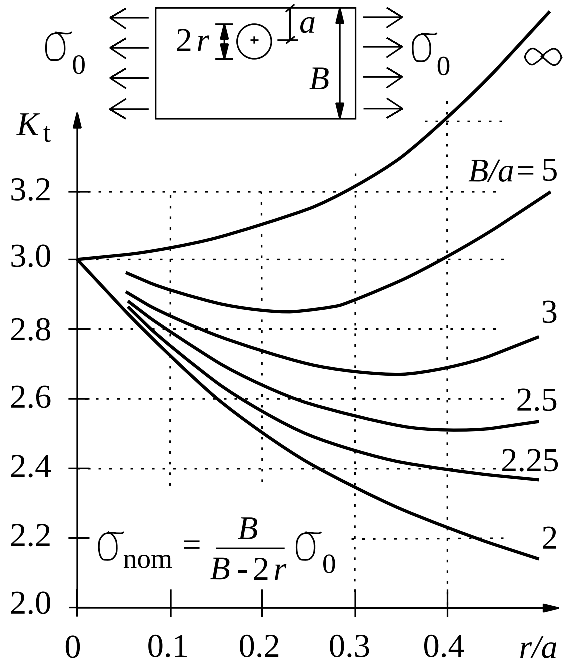
Tension of flat bar with notch



Tension of circular bar with shoulder fillet



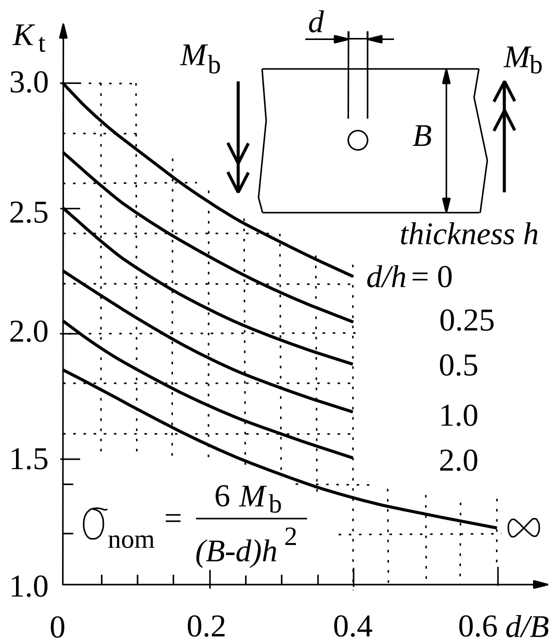
Tension of circular bar with U-shaped groove



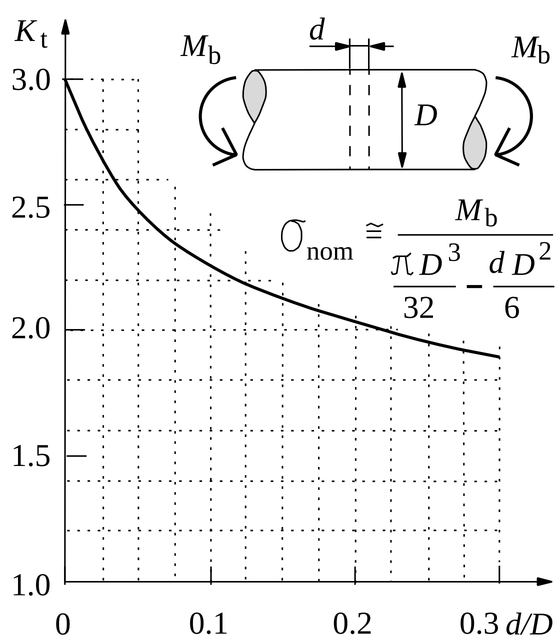
Tension of flat bar with hole

Bending

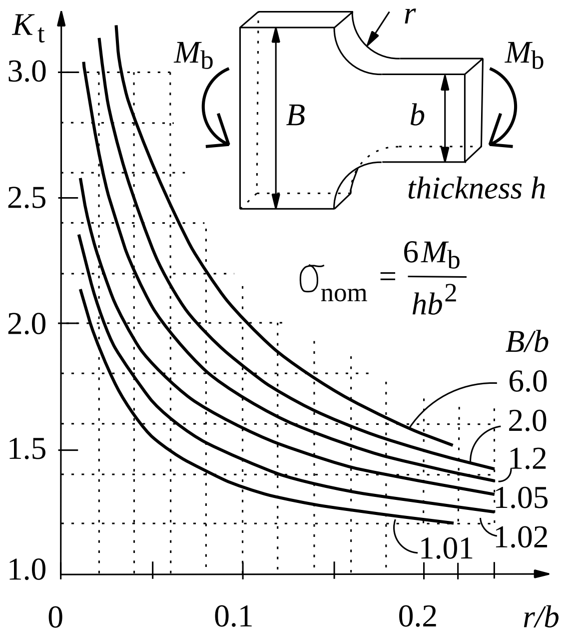
Maximum normal stress at a stress concentration is $\sigma_{max} = K_t \sigma_{nom}$, where K_t and σ_{nom} are given in the diagrams



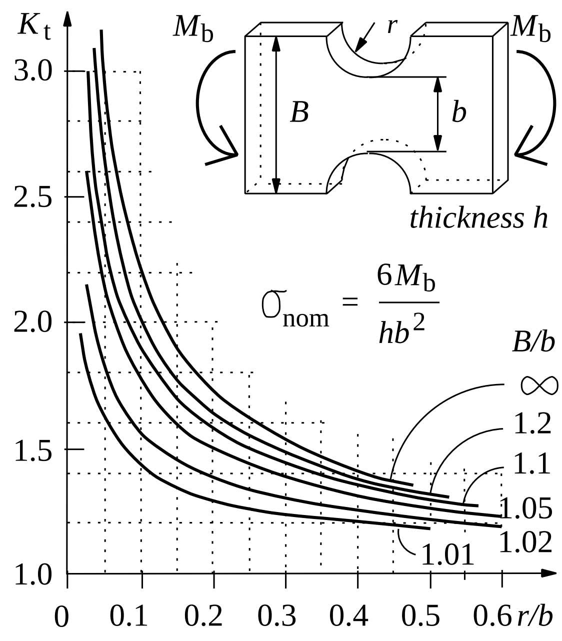
Bending of flat bar with hole



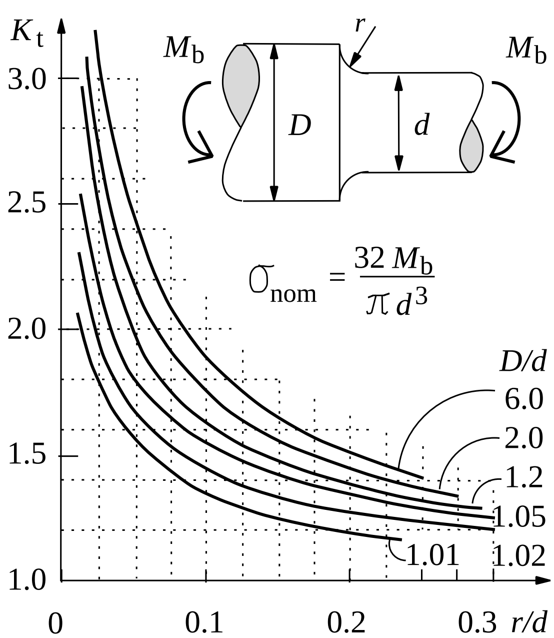
Bending of circular bar with hole



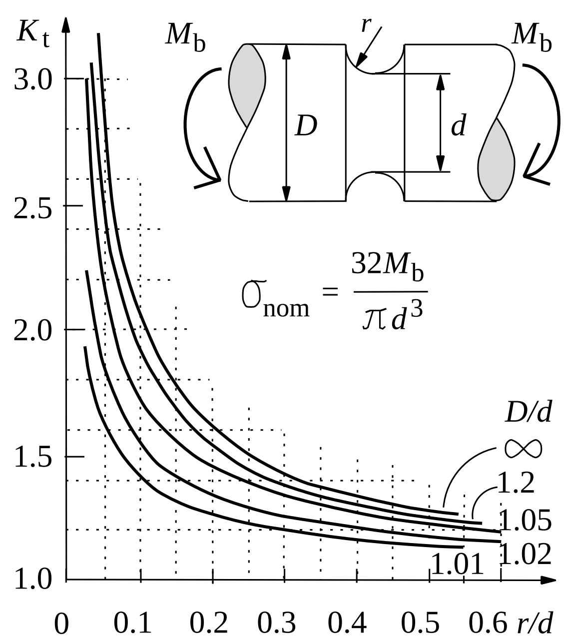
Bending of flat bar with shoulder fillet



Bending of flat bar with notch



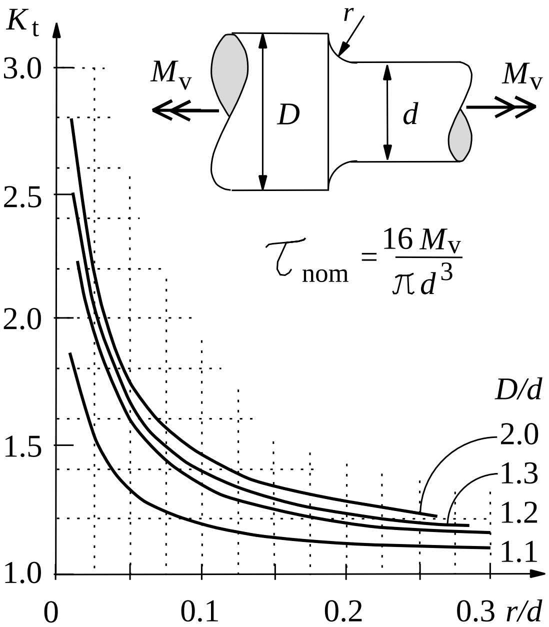
Bending of circular bar with shoulder fillet



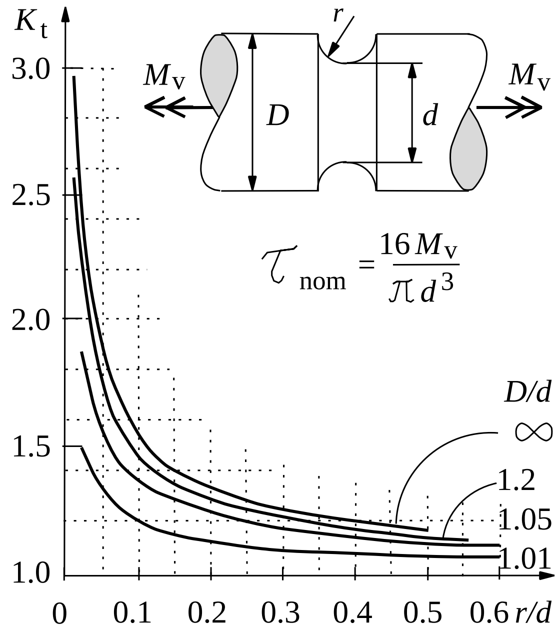
Bending of circular bar with U-shaped groove

Torsion

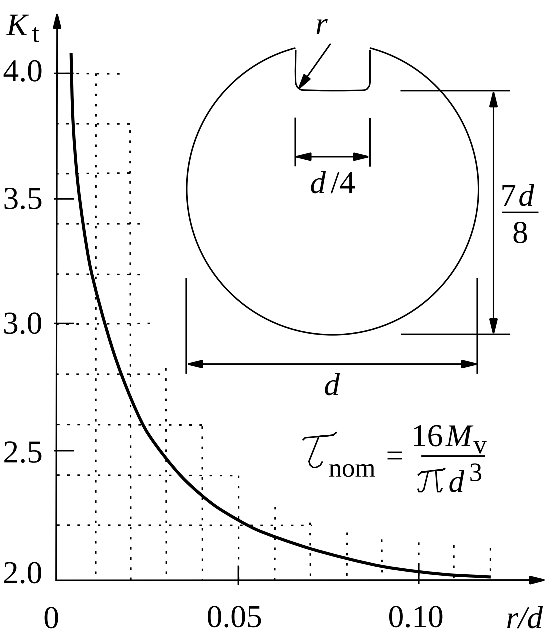
Maximium shear stress at stress concentration is $\tau_{max} = K_t \tau_{nom}$, where K_t and τ_{nom} are given in the diagrams



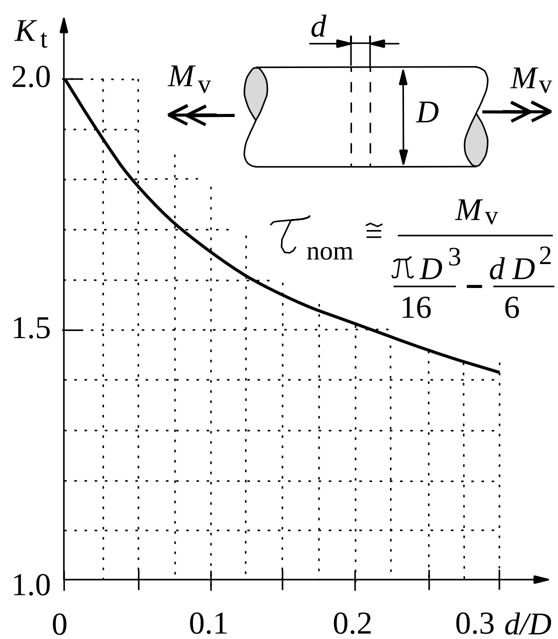
Torsion of circular bar with shoulder fillet



Torsion of circular bar with notch



Torsion of bar with longitudinal keyway



Torsion of circular bar with hole

10. Material data

The following material properties may be used *only* when solving exercises. For a real design, data should be taken from latest official standard and *not* from this table (two values for the same material means different qualities).[1]

Material	Young's modulus E GPa	ν –	$\alpha 10^6$ K^{-1}	Ultimate strength MPa	Yield limit tension/ compression MPa	bending MPa	torsion MPa
Carbon steel							
141312-00	206	0.3	12	360 460	>240	260	140
141450-1	205	0.3		430 510	>250	290	160
141510-00	205	0.3		510 640	>320		
141550-01	205	0.3		490 590	>270	360	190
141650-01	206	0.3	11	590 690	>310	390	220
141650	206	0.3		860	>550	610	
					Offset yield strength $R_{p0.2}$ ($\sigma_{0,2}$)		
Stainless steel							
2337-02	196	0.29	16.8	>490	>200		
Aluminium							
SS 4120-02	70		23	170 215	>65		
SS 4120-24	70		23	220 270	>170		
SS 4425-06	70		23	>340	>270		

[1] Data in this table has been collected from B Sundström (editor): Handbok och Formelsamling i Hållfasthetslära, Institutionen för hållfasthetslära, KTH, Stockholm, 1998.